THE POSSIBILITY OF MEASURING TURBULENCE BY A CORRELATION METHOD

N. F. Derevyanko

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In [1] a correlation method for measuring the velocity pulsations in stationary plasma flows was described. The magnitude of the pulsations was determined from the value of the frequency deviation in the spectrum of the cross-correlation functions of optical fluctuations at two closely arranged points along the flow.

In the present work, an attempt is made to justify such a method for measuring the characteristics of turbulence both in plasma and in any low-temperature gas flows.

1. The effect of turbulent velocity pulsations on the frequency spectrum of optical fluctuations. We assume that there exists a homogeneous isotropic random field M, the elements of which are random sets of hydrodynamic characteristics of a turbulent flow of incompressible liquid .

We register a field of optical fluctuations N, assuming that N is the image of the field M for the mapping g; i.e.,

$$g(M) = \{g(X);\ X \in M\}. \tag{1.1}$$

Here the element X is a random set.

At the same time the mapping is assumed to be mutually single-value. That is, if $g: M \to N$, then for any $\alpha \in g(M)$ there exists only one element $X \in M$ such that $g(X) = \alpha$ (α is a random quantity serving as an element of the field N).

In this case the field N is also homogeneous and isotropic.

The assumptions formulated above must in fact be kept in mind for all optical methods of turbulence measurement.

As is known [2], the condition of homogeneity of a random field indicates that its mean value is constant, while the correlation function does not vary for a simultaneous displacement of the pair of point $\mathbf{r}_1$ and $\mathbf{r}_2$ in the same direction by the same amount; i.e.,

$$\langle \overline{N(\mathbf{r})} \rangle = \text{const},\quad B_N(\mathbf{r}_1, \mathbf{r}_2) = B(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.2}$$

The condition of isotropy, in addition, requires that $B_N(\mathbf{r})$ depend only on $r = |\mathbf{r}|$, i.e., only on the distance between the points of observation, while the spectral density, consequently, would be a function of only one variable, the modulus of the vector $\varkappa$:

$$\Phi(\varkappa) = \frac{1}{2\pi^2\varkappa} \int_0^\infty B_N(r) \sin \varkappa r dr. \tag{1.3}$$

The three-dimensional spectral density $\Phi(\varkappa)$ of an isotropic random field is related to the one-dimensional spectral density $S_N(\varkappa)$ by the simple equation

$$\Phi(\varkappa) = -\frac{1}{2\pi\varkappa} \frac{dS_N(\varkappa)}{d\varkappa}. \tag{1.4}$$

In going over from spatial spectra to time spectra for measurements at a point we often employ a "frozen turbulence" model [2–4]. This means that we assume all time variations of $N(\mathbf{r}, t)$ are associated with the transfer of a spatial distribution of the field with a constant velocity $\mathbf{v}$. At the same time the transfer takes place without any evolution:

$$B(\mathbf{r}, \tau) = B(\mathbf{r} - \mathbf{v}\tau), \tag{1.5}$$

$$\Phi(\varkappa) = -\frac{v^2}{2\pi\varkappa} W'(\varkappa v), \tag{1.6}$$

where $W(\varkappa v)$ is the time spectrum of the isotropic "frozen" field.

Accordingly, the homogeneous spatial spectral density (for example, the component N_x of the field) is connected with the time spectrum at the point A, lying on the x-axis, by the relation

$$S_{N_x}(\varkappa) = uW_A(u\varkappa). \tag{1.7}$$

From the example of expression (1.7) we see clearly that when the conditions of "being frozen" are not satisfied, the frequency spectrum at the point will be displaced along the frequency scale and will be deformed as a function of the transfer velocity for the spatial distribution of the field.

Therefore, the following step, it seems to us, involves a model according to which the transfer of the spatial distribution of the field is carried out without evolution, but with a velocity which varies with time. In this case

$$N(\mathbf{r}, t + t') = N\left(\mathbf{r} - \int_0^{t'} \mathbf{v}(t)\, dt,\ t\right),$$

$$B_N(\mathbf{r}, \tau) = \langle N(\mathbf{r} + \mathbf{r}_1, t + \tau) N(\mathbf{r}_1, t) \rangle =$$

$$= \left\langle N\left(\mathbf{r} + \mathbf{r}_1 - \int_0^{t+\tau} \mathbf{v}(t)\, dt,\ 0\right) N\left(\mathbf{r}_1 - \int_0^{t} \mathbf{v}(t)\, dt,\ 0\right)\right\rangle =$$

$$= B_N\left(\mathbf{r} - \int_0^{\tau} \mathbf{v}(\tau)\, d\tau\right)$$

i.e., for a "frozen" field moving with variable velocity v(t), the relation

$$B_N(\mathbf{r}, t) = B_N\left(\mathbf{r} - \int_0^{\tau} \mathbf{v}(\tau)\, d\tau\right) \tag{1.8}$$

is satisfied, or, for the component of the field along the x-axis, we have

$$B(x, t) = B\left(x - \int_0^{\tau} u(\tau)\, d\tau\right). \tag{1.9}$$

For a homogeneous random field, the integral canonical representation of the spatial correlation function for the component N_x has the form

$$B_{N_x}(r - r') = \int_{-\infty}^{\infty} S_N(\varkappa)\, e^{i\varkappa r} \overline{e^{i\varkappa r'}}\, d\varkappa. \tag{1.10}$$

The integral canonical representation of the stationary random function $N_x(r)$ corresponds to the following:

$$N_x(r) = m_{N_x} + \int_{-\infty}^{\infty} V(\varkappa)\, e^{i\varkappa r}\, d\varkappa. \tag{1.11}$$

Placing $r = \int_0^t u(t)\, dt$ in (1.10), for the case of a "frozen" field moving with a varying time-velocity, we have at the given point

$$N_x(t) = m_{N_x} + \int_{-\infty}^{\infty} V\left(\frac{2\pi}{\lambda}\right) \exp\left(i \frac{2\pi}{\lambda} \int_0^t u(t)\, dt\right) d\left(\frac{2\pi}{\lambda}\right), \tag{1.12}$$

where $\lambda = 2\pi/\varkappa$.

What is the physical meaning of this expression? If $u(t) = \langle u \rangle + u'(t)$, where $u'(t)$ is the centered stationary random function,

$$N_x(t) = m_{N_x} + \int_{-\infty}^{\infty} V\left(\frac{2\pi}{\lambda}\right) \exp\left\{i \frac{2\pi}{\lambda} \left[\langle u \rangle t + \int_0^t u'(t)\, dt\right]\right\} d\left(\frac{2\pi}{\lambda}\right) =$$

$$= m_{N_x} + \int_{-\infty}^{\infty} V\left(\frac{\omega_0}{\langle u \rangle}\right) \frac{1}{\langle u \rangle} \exp\left\{i\left(\omega_0 t + \int_0^t \omega(t)\, dt\right)\right\} d\omega_0. \tag{1.13}$$

That is to say, $N_x(t)$ is a result of the frequency modulation of the random functions $\varepsilon(t)$ by the random process $u(t)$:

$$\varepsilon(t) = m_\varepsilon + \int_{-\infty}^{\infty} V\left(\frac{\omega_0}{\langle u \rangle}\right) \frac{1}{\langle u \rangle}\, e^{i\omega_0 t}\, d\omega_0. \tag{1.14}$$

The latter is interpreted as the component N_x of the field when the "frozen" field is being displaced with a constant velocity $\langle u\rangle$. In the presence of velocity pulsations, it is modulated in frequency.

We rewrite (1.13) in the form

$$N_x(t) = m_{N_x} + \int_{-\infty}^{\infty} V\left(\frac{2\pi}{\lambda}\right) \exp\left\{i\frac{2\pi}{\lambda}\left[\langle u\rangle t + \int_0^t u'(t)\,dt\right]\right\} d\left(\frac{2\pi}{\lambda}\right) =$$

$$= m_{N_x} + \int_{-\infty}^{\infty} V\left(\frac{2\pi}{\lambda}\right) \exp\left\{i\frac{2\pi}{\lambda}\left[\langle\xi\rangle + \xi'\right]\right\} d\left(\frac{2\pi}{\lambda}\right). \qquad (1.15)$$

Here $\langle\xi\rangle$ is the coordinate of the component N_x of the field at the instant t for transfer with constant velocity; ξ' is the pulsation of this coordinate for motion with variable velocity.

To find the dispersion $\langle\xi^2(t)\rangle$ we can use the basic formula of the theory of turbulent diffusion [4]:

$$\langle\xi^2(t)\rangle = 2K_u(0)\int_0^t (t-\tau)\,R_u(\tau)\,d\tau\,. \qquad (1.16)$$

Here $K_u(0)$ is the velocity dispersion; $R_u(\tau)$ is the correlation coefficient between velocity pulsations at different instants of time.

From expression (1.16) it follows that the amplitude of the modulating function is the mean square velocity pulsation. This means that the frequency deviation of each elemental harmonic of the process E(t) can be represented according to the formula

$$\Delta\omega_v = \frac{2\pi}{\lambda_v}\sqrt{K_u(0)} = \frac{\omega_v}{\langle u\rangle}\sqrt{K_u(0)}\,. \qquad (1.17)$$

Let us consider (1.16) for large t, so that $t \gg t^*$. Beginning with the instant t^*, $R_u(\tau)$ is everywhere approximately equal to zero.

We then have

$$\langle\xi^2(t)\rangle = \lim_{T\to\infty} 2K_u(0)(t)\int_0^{\infty} R_u(\tau)\,d\tau = \lim_{T\to\infty} 2K_u(0)\,t\,\mathcal{T}_E\,. \qquad (1.18)$$

where $\mathcal{T}_E$ is the Euler integral time scale.

From (1.18) we find a quantity that is reciprocal to the integral time scale:

$$\frac{1}{\mathcal{T}_E}\left[\frac{1}{T}\right] = \frac{2K_u(0)\,t}{\langle\xi^2(t)\rangle}\left[\frac{U^2T}{U^2T^2}\right]. \qquad (1.19)$$

The first part of (1.19) constitutes the mean-square frequency pulsation, i.e., the working frequency of modulation. Thus, the frequency of modulation is directly determined by the integral time scale:

$$\Omega_m = 2\pi\frac{1}{\mathcal{T}_E}\,, \qquad \mathcal{T}_E = \frac{2\pi}{\Omega_m}\,. \qquad (1.20)$$

This has a definite physical meaning, since the magnitude of the integral time scale can serve as a measure of the longest time interval during which the spatial distribution of the field on the average is transferred in a given direction. In other words, it is the maximum modulation period.

From relation (1.20) it follows that the larger the integral time scale, the smaller the working frequency of modulation. In the limiting case, when $\mathcal{T}_E = \infty$, $\Omega_m = 0$. This corresponds to the absence of modulation (transfer of the "frozen" field with a constant velocity).

Conversely, when $\mathcal{T}_E \to 0$, $\Omega_m \to \infty$; i.e., when the character of the velocity pulsations approaches white noise, the frequency of modulation becomes infinitely high.

Thus, the random function $N_x(t)$ carries in itself information about the turbulence in the form of a modulating function.

Since the modulation index, given by the intensity of turbulence, is usually much less than unity, it is of interest to consider the FM (frequency modulation) problem with a small index of one random process by another.

2. **Frequency modulation with a small index of a random process ε(t) by the random process u(t).** We assume that there exists a stationary random function ε(t) specified in the canonical representation:

$$\varepsilon(t) = m_\varepsilon + \int_{-\infty}^{\infty} V(\omega)\,e^{i\omega t}\,d\omega\,. \qquad (2.1)$$

For the sake of simplicity, we set $m_\varepsilon = 0$.

We transform the process ε(t) in such a way that the process u(t), being also a stationary random function, would be the modulating function with respect to the frequency of each elemental harmonic of ε(t).

If $u(t) = \sum_\Omega V(\Omega)e^{i\Omega t}$, while $\langle u(t)\rangle = 0$, the frequency of each elemental harmonic of the process ε(t) varies according to the law

$$\omega_1(t) = \omega_j + \sum_\Omega V(\Omega)\,e^{i\Omega t}\,.$$

The elementary function assumes the form

$$y_j = V(\omega_j)\cos\int_0^t \omega_j(t)\,dt + iV(\omega_j)\sin\int_0^t \omega_j(t)\,dt =$$

$$= V(\omega_j)\exp\left\{i\int_0^t \omega_j(t)\,dt\right\} \qquad (2.2)$$

or

$$y_j = V(\omega_j)\cos\left[\omega_j t + \int_0^t \sum_\Omega V(\Omega)\,e^{i\Omega t}\,dt\right] +$$

$$+ iV(\omega_j)\sin\left[\omega_j t + \int_0^t \sum_\Omega V(\Omega)\,e^{i\Omega t}\,dt\right] =$$

$$= V(\omega_j)\cos\left[\omega_j t + \sum_\Omega \frac{V(\Omega)}{i\Omega}\,e^{i\Omega t}\right] +$$

$$+ iV(\omega_j)\sin\left[\omega_j t + \sum_\Omega \frac{V(\Omega)}{i\Omega}\,e^{i\Omega t}\right]. \qquad (2.3)$$

Finding the cosine and sine of the sum, we obtain

$$y_j = V(\omega_j)\left[\cos\omega_j t\cos\sum_\Omega \beta e^{i\Omega t} - \sin\omega_j t\sin\sum_\Omega \beta e^{i\Omega t}\right] +$$

$$+ iV(\omega_j)\left[\sin\omega_j t\cos\sum_\Omega \beta e^{i\Omega t} + \cos\omega_j t\sin\sum_\Omega \beta e^{i\Omega t}\right].$$

For small β we obtain

$$y_j = V(\omega_j)\left[\cos\omega_j t - \sum_\Omega \beta e^{i\Omega t}\sin\omega_j t\right] +$$

$$+ iV(\omega_i)\left[\sin\omega_j t + \sum_\Omega \beta e^{i\Omega t}\cos\omega_j t\right]. \qquad (2.4)$$

We sum over all elemental harmonics:

$$x(t) = \int_{-\infty}^{\infty} V(\omega_j)\cos\omega_j t\,d\omega_j - \int_{-\infty}^{\infty} V(\omega_j)\sum_\Omega \beta e^{i\Omega t}\sin\omega_j t\,d\omega_j +$$

$$+ \int_{-\infty}^{\infty} iV(\omega_j)\sin\omega_j t\,d\omega_j + \int_{-\infty}^{\infty} iV(\omega_j)\sum_\Omega \beta e^{i\Omega t}\cos\omega_j t\,d\omega_j =$$

$$= \int_{-\infty}^{\infty} V(\omega_j)\,e^{i\omega_j t}\,d\omega_j + \int_{-\infty}^{\infty} V(\omega_j)\sum_\Omega \beta e^{i\Omega t}e^{-i\Omega t}\,d\omega_j =$$

$$= \varepsilon(t) + \sum_\Omega \beta e^{i\Omega t}\varepsilon(t) = \varepsilon(t)\left[1 + \sum_\Omega \frac{V(\Omega)}{i\Omega}\,e^{i\Omega t}\right] =$$

$$= \varepsilon(t)\,[1 + u^*(t)]\,. \qquad (2.5)$$

Consequently, frequency modulation with a small index of one random process by another yields the product of two random functions.

3. **The statistical spectrum of the record of optical fluctuations at a fixed point.** We rewrite (2.5) in the form

$$x(t) = Z(t)\,\varepsilon(t)\,,$$

where $Z(t) = 1 + u^*(t)$.

If the processes Z(t) and ε(t) are independent, as is well known [5], the correlation function of the product of these random processes equals the product of their correlation functions; i.e.,

$$K_{xx}(\tau) = K_{zz}(\tau)\,K_{\varepsilon\varepsilon}(\tau)\,, \qquad (3.1)$$

where

$$K_{zz}(\tau) = M\{[1 + u^*(t)][1 + u^*(t')]\} =$$
$$= M\{1 + u^*(t') + u^*(t) + u^*(t)u^* \times$$
$$\times (t')\} = 1 + K_{u^*u^*}(\tau)$$

for

$$M\{u^*(t')\} = M\{u^*(t)\} = 0, \quad t' = t + \tau .$$

Thus,

$$K_{xx}(\tau) = K_{\varepsilon\varepsilon}(\tau) + K_{u^*u^*}(\tau) K_{\varepsilon\varepsilon}(\tau) . \tag{3.2}$$

Hence,

$$S_x(\omega) = S_\varepsilon(\omega) + \frac{1}{\pi} \int_{-\infty}^{\infty} S_{u^*}(\Omega) S_\varepsilon(\omega - \Omega)\, d\Omega . \tag{3.3}$$

We assume that the functions of the spectral density of the processes $\varepsilon(t)$ and $u^*(t)$ have the form of Gaussian curves; i.e.,

$$S_\varepsilon(\omega) = \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp - \left\{ \frac{(\omega - \langle\omega\rangle)^2}{2\sigma_\omega^2} \right\} S_{u^*}(\omega) =$$
$$= \frac{1}{\sigma_\Omega \sqrt{2\pi}} \exp \left\{ - \frac{(\Omega - \langle\Omega\rangle)^2}{2\sigma_\Omega^2} \right\} . \tag{3.4}$$

Then

$$S_x(\omega) = \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp \left[- \frac{(\omega - \langle\omega\rangle)^2}{2\sigma_\omega^2} \right] +$$
$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma_\Omega \sqrt{2\pi}} \exp \left[- \frac{(\Omega - \langle\Omega\rangle)^2}{2\sigma_\Omega^2} \right] \times$$
$$\times \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp \left[- \frac{[(\omega - \omega) - \Omega]^2}{2\sigma_\omega^2} \right] d\Omega . \tag{3.5}$$

The spectral density given by the first term of the right-hand side of (3.5) is maximum for $\omega = \langle\omega\rangle$ and is symmetric about it.

Let us investigate the behavior of the spectral density given by the second term of the right-hand side of (3.5) for ω close to $\langle\omega\rangle$.

At the point $\omega = \langle\omega\rangle$ this term equals ($\langle\Omega\rangle = 0$ for the sake of simplicity)

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_\Omega \sqrt{2\pi}} \exp \left[- \frac{\Omega^2}{2\sigma_\Omega^2} \right] \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp \left[- \frac{\Omega^2}{2\sigma_\omega^2} \right] d\Omega = \right.$$
$$\left. = \frac{1}{2\pi^2 \sigma_\Omega \sigma_\omega} \int_{-\infty}^{\infty} \exp \left[- \Omega^2 \left(\frac{1}{2\sigma_\Omega^2} + \frac{1}{2\sigma_\omega^2} \right) \right] \right] d\Omega = 0 .$$

Thus, for $\omega = \langle\omega\rangle$ and $\omega = \infty$, the spectral density given by the second term decreases to zero.

It is obvious that approaching $\omega = \langle\omega\rangle$ from either positive or negative frequencies ω allows us to obtain the same reduction to zero in the spectral density.

The first term of spectral density (3.5) is represented in Fig. 1 by the curve a. The next term is the curve b.

In Fig. 2 (a, b, c) we have illustrated examples of the resulting spectral functions (in the order of rising mathematical expectation of modulating function frequency).

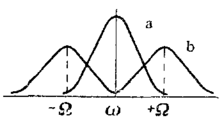

Fig. 1

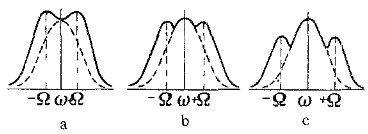

Fig. 2

If we take into account the fact that the function of spectral density $S_x(\omega)$ repeats the law of distribution of the probability density with respect to frequencies, then the value of the effective deviation $\Delta\omega_{\text{eff}}$ can be calculated from the formula

$$\Delta\omega_{\text{eff}} = \frac{\int_{-\infty}^{\infty} (\omega - \langle\omega\rangle)^2 S_x(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_x(\omega)\, d\omega} . \tag{3.6}$$

The latter, in turn, is connected with the value of the mean square pulsation of the velocity by relation (1.17).

4. **Conclusion.** Thus, on the basis of the model of a "frozen" field being transferred with a time-varying velocity, a correlation method can be used to measure the turbulence.

The model of a "frozen" field being transferred with a time-varying velocity, in the same way as was the first model of a "frozen" field, is valid in real turbulent media only within the limits of the turbulence scale. Indeed, within the limitation $l \ll L_0$ we can assume that the velocity at all points of the field is the same and only varies with time.

However, if, for the realization of the first model, we must resort to method of structural functions, then, in the second model, their use cannot give the required result.

It is not difficult to show that if at the first point we write the function

$$X_1(t) = [1 + u^* \times (t)]\, \varepsilon(t),$$

while at the second, located by the distance $l \ll L_0$ downstream, we write

$$X_2(t) = [1 + u^*(t + \varphi)]\, \varepsilon(t),$$

then the structural function $D(x, t)$ will not contain, in explicit form, any information about the turbulent velocity pulsations.

At the same time the cross-correlation function, suppressing the effect of noise, reproduces the required frequency-modulated signal [1].

The spectrum of the cross-correlation function contains information about the magnitude of the longitudinal component of turbulent velocity pulsations and about the spectra of the random process $\varepsilon(t)$ being modulated and the random modulating process $u(t)$.

Thus, the use of structural functions must define more accurately the boundary of the spectrum of the process being modulated. Hence, the combined use of structural functions side by side with cross-correlation functions may be advisable.

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